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New Seven-Step Numerical Method for Direct Solution of Fourth Order Ordinary Differential Equations

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Abstract: A new numerical method for solving fourth order ordinary differential equations directly is proposed in this paper. Interpolation and collocation were employed in developing this method using seven steps. The use of the approximated power series as an interpolation equation was adopted in deriving the method. The basic properties of the new method such as zero-stability, consistency, convergence and order are established. The numerical results indicate that the new method gives better accuracy than the existing methods when it is applied to fourth ordinary differential equations.

Keywords: *block method; collocation; direct solution; interpolation; ordinary differential equations; seven-step.*

1 Introduction

The general fourth order ordinary differential equations (ODEs) of the form

$$y^{(4)} = f(x, y, y', y'', y''') \quad y^{(s)}(x_0) = y_s, s = 0(1)3, x \in [x_0, b] \tag{1}$$

are considered in this paper. Eq. (1) can be solved by reducing it to its equivalent first order system as mentioned in [1-9]. However, this approach suffers some setbacks, such as evaluation of too many functions and heavy computation (see [10-12]).

Direct methods of solving Eq. (1) have been examined by several researchers [10,12-16]. They developed linear multistep methods using interpolation and collocation whereby the use of the approximated power series as a basis function was considered. Kayode [12] developed an efficient zero-stable numerical method with step number k = 4 and 5 for fourth order initial value problems (IVPs), which was implemented in predictor-corrector mode. Furthermore, a five-step block method for solving fourth order ODEs directly is presented in [10]. In addition, in [13] and [14] six-step block methods are developed for solving fourth order ODEs using a multistep collocation approach whereby collocation points are selected at some grid points. These methods, however, have low accuracy.

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In order to improve the accuracy of the existing methods, this article proposes a new block method for directly solving general fourth order IVPs of ODEs by increasing step number *k*.

2 Methodology

Let the approximate solution to Eq. (1) be a power series of the form

$$y(x)=\sum_{j=0}^{k+4} a_j x^j \tag{2}$$

where *k* is the step number. The fourth derivative of Eq. (2) gives

$$y^{(4)}(x)=\sum_{j=4}^{k+4} j(j-1)(j-2)(j-3)a_j x^{j-4} = f(x,y,y',y'',y''') \tag{3}$$

Eq. (2) is interpolated at $x=x_{n+i}, i=(k-5)(1)(k-2)$ and Eq. (3) is collocated at $x=x_{n+i}, i=0(1)k$. This gives a system of equations in the form

$$AX=B \tag{4}$$

where

$$X=\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{k+4} \end{bmatrix},\ B=\begin{bmatrix} y_{n+k-5} \\ y_{n+k-4} \\ y_{n+k-3} \\ y_{n+k-2} \\ \vdots \\ y_{n+k} \end{bmatrix},$$

$$A=\begin{pmatrix} 1 & x_{n+k-5} & x_{n+k-5}^2 & x_{n+k-5}^3 & x_{n+k-5}^4 & x_{n+k-5}^4 & \ldots\ldots & x_{n+k-5}^{k+4} \\ 1 & x_{n+k-4} & x_{n+k-4}^2 & x_{n+k-4}^3 & x_{n+k-4}^4 & x_{n+k-4}^4 & \ldots\ldots & x_{n+k-4}^{k+4} \\ 1 & x_{n+k-3} & x_{n+k-3}^2 & x_{n+k-3}^3 & x_{n+k-3}^4 & x_{n+k-3}^4 & \ldots\ldots & x_{n+k-3}^{k+4} \\ 1 & x_{n+k-2} & x_{n+k-2}^2 & x_{n+k-2}^3 & x_{n+k-2}^4 & x_{n+k-2}^4 & \ldots\ldots & x_{n+k-2}^{k+4} \\ 0 & 0 & 0 & 0 & 24 & 120x_n & \ldots\ldots & (k+4)(k+3)(k+2)(k+1)x_n^k \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+1} & \ldots\ldots & (k+4)(k+3)(k+2)(k+1)x_{n+1}^k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+k} & \ldots\ldots & (k+4)(k+3)(k+2)(k+1)x_{n+k}^k \end{pmatrix}$$

By using Gaussian elimination, the values of $a_j's, j = 0(1)k+4$ in Eq. (4) are obtained and then substituted into Eq. (2) to give a continuous linear multistep method in the form

$$y(t) = \sum_{j=k-5}^{k-2} \alpha_j(t) y_{n+j} + h^4 \sum_{j=0}^{k} \beta_j(t) f_{n+j} \tag{5}$$

For $k = 7$, we have

$$\begin{pmatrix} \alpha_2(t) \\ \alpha_3(t) \\ \alpha_4(t) \\ \alpha_5(t) \end{pmatrix} = \begin{pmatrix} -1 & \frac{-11}{6} & -1 & \frac{-1}{6} \\ 4 & 7 & \frac{7}{2} & \frac{1}{2} \\ -6 & \frac{-19}{2} & -4 & \frac{-1}{2} \\ 4 & \frac{13}{3} & \frac{3}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \end{pmatrix}$$

$$\begin{pmatrix} \beta_0(t) \\ \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \\ \beta_4(t) \\ \beta_5(t) \\ \beta_6(t) \\ \beta_7(t) \end{pmatrix} = G \begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \\ t^9 \\ t^{10} \\ t^{11} \\ t^{12} \end{pmatrix}$$

where

$$G=\begin{pmatrix} 0 & \frac{-360}{w} & \frac{-27918}{w} & \frac{-41261}{w} & 0 & \frac{23760}{w} & \frac{10164}{w} & \frac{-1386}{w} & \frac{-1980}{w} & \frac{-550}{w} & \frac{-66}{w} & \frac{-3}{w} \\ 0 & \frac{66804}{w} & \frac{264330}{w} & \frac{353342}{w} & 0 & \frac{-199584}{w} & \frac{-83160}{w} & \frac{12672}{w} & \frac{16335}{w} & \frac{4345}{w} & \frac{495}{w} & \frac{21}{w} \\ \frac{-44880}{u} & \frac{-73220}{u} & \frac{-130768}{u} & \frac{-151569}{u} & 0 & \frac{83160}{u} & \frac{33264}{u} & \frac{-5874}{u} & \frac{-6600}{u} & \frac{-1650}{u} & \frac{-176}{u} & \frac{-7}{u} \\ \frac{4243536}{v} & \frac{7690860}{v} & \frac{4523310}{v} & \frac{1277606}{v} & 0 & \frac{-332640}{v} & \frac{-123816}{v} & \frac{26928}{v} & \frac{24849}{v} & \frac{5665}{v} & \frac{561}{v} & \frac{21}{v} \\ \frac{15608736}{v} & \frac{30203040}{v} & \frac{17327970}{v} & \frac{2414269}{v} & 0 & \frac{498960}{v} & \frac{158004}{v} & \frac{-47718}{v} & \frac{-32076}{v} & \frac{-6490}{v} & \frac{-594}{v} & \frac{-21}{v} \\ \frac{2357520}{u} & \frac{6661340}{u} & \frac{6852142}{u} & \frac{2812194}{u} & 0 & \frac{-332640}{u} & \frac{-49896}{u} & \frac{30624}{u} & \frac{14025}{u} & \frac{2475}{u} & \frac{209}{u} & \frac{7}{u} \\ \frac{-403920}{w} & \frac{619596}{w} & \frac{4781700}{w} & \frac{7364533}{w} & \frac{4989600}{w} & \frac{1446984}{w} & \frac{-64680}{w} & \frac{-176022}{w} & \frac{-55440}{w} & \frac{-8470}{w} & \frac{-660}{w} & \frac{-21}{w} \\ \frac{39600}{w} & \frac{31380}{w} & \frac{-130878}{w} & \frac{-185614}{w} & 0 & \frac{142560}{w} & \frac{116424}{w} & \frac{45936}{w} & \frac{10395}{w} & \frac{1375}{w} & \frac{99}{w} & \frac{3}{w} \end{pmatrix}.$$

The values of w, u and v are w = 119750400, u = 13305600, v = 23950080, for $t=\frac{x-x_{n+6}}{h}$.

Eq. (5) is evaluated at the non-interpolating points, i.e. t = -6, -5, 0 and 1, to produce the discrete schemes. The first, second and third derivatives of Eq. (5) are evaluated at all the points within the interval, i.e. t = -6, -5, -4, -3, -2, -1, 0 and 1, to give the derivatives of the discrete schemes. These schemes are combined in a matrix, whereby both the y and the f function are multiplied by the inverse of the coefficients of $y_{n+j}, j=0(1)k$. This yields a block of the form

$$A^0Y_N = A'Y_{N-1} + hA''Y'_{N-1} + h^2B'Y''_{N-1} + h^3B''Y'''_{N-1} + h^4(E^0F_N + E^1F_{N-1}) \qquad (6)$$

where

$$Y_N = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{n+k} \end{pmatrix}, Y_{N-1} = \begin{pmatrix} y_{n-k+1} \\ y_{n-k+2} \\ \vdots \\ y_n \end{pmatrix} \; Y'_{N-1} = \begin{pmatrix} y'_{n-k+1} \\ y'_{n-k+2} \\ \vdots \\ y'_n \end{pmatrix}, \; Y''_{N-1} = \begin{pmatrix} y''_{n-k+1} \\ y''_{n-k+2} \\ \vdots \\ y''_n \end{pmatrix},$$

$$Y'''_{N-1} = \begin{pmatrix} y'''_{n-k+1} \\ y'''_{n-k+2} \\ \vdots \\ y'''_n \end{pmatrix}, F_N = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ \vdots \\ f_{n+k} \end{pmatrix}, \; F_{N-1} = \begin{pmatrix} f_{n-k+1} \\ f_{n-k+2} \\ \vdots \\ f_n \end{pmatrix}.$$

If k = 7, we obtain

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A'' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{49}{2} \end{pmatrix},$$

$$B'' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{32}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{125}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 36 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{343}{6} \end{pmatrix}, \quad E^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{3102701}{119750400} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{315461}{467775} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{526077}{492800} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{312608}{467775} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{25842625}{4790016} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{110052}{11550} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{262892693}{17107200} \end{pmatrix},$$

$$E^0 = \begin{pmatrix} \frac{4137616}{119750400} & -\frac{5122521}{119750400} & \frac{5415020}{119750400} & -\frac{3967805}{119750400} & \frac{1878984}{119750400} & -\frac{517351}{119750400} & \frac{62956}{119750400} \\ \frac{315461}{467775} & -\frac{306810}{467775} & \frac{320335}{467775} & -\frac{233050}{467775} & \frac{109899}{467775} & -\frac{30176}{467775} & \frac{3665}{467775} \\ \frac{1616436}{492800} & -\frac{1195317}{492800} & \frac{1348200}{492800} & -\frac{985365}{492800} & \frac{465588}{492800} & -\frac{127971}{492800} & \frac{15552}{492800} \\ \frac{1096960}{467775} & -\frac{616032}{467775} & \frac{838400}{467775} & -\frac{599480}{467775} & \frac{283392}{467775} & -\frac{77920}{467775} & \frac{9472}{467775} \\ \frac{98195000}{4790016} & -\frac{42733125}{4790016} & \frac{74762500}{4790016} & -\frac{49290625}{4790016} & \frac{23688000}{4790016} & -\frac{6516875}{4790016} & \frac{792500}{4790016} \\ \frac{440802}{11550} & -\frac{151632}{11550} & \frac{344790}{11550} & -\frac{199260}{11550} & \frac{103518}{11550} & -\frac{27972}{11550} & \frac{3402}{11550} \\ \frac{1093194508}{17107200} & -\frac{301769685}{17107200} & \frac{884720480}{17107200} & -\frac{438242525}{17107200} & \frac{266827932}{17107200} & -\frac{64354003}{17107200} & \frac{8163400}{17107200} \end{pmatrix}$$

The corresponding derivatives of Eq. (6) are given by

$$\begin{pmatrix} y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \\ y'_{n+4} \\ y'_{n+5} \\ y'_{n+6} \\ y'_{n+7} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} y'_n + \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{pmatrix} hy'' + \begin{pmatrix} \frac{1}{2} \\ 2 \\ \frac{9}{2} \\ 8 \\ \frac{25}{2} \\ 18 \\ \frac{49}{2} \end{pmatrix} h^2 y''' + h^3 T \begin{pmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \end{pmatrix}$$

where

$$T = \begin{pmatrix} \frac{335799}{3628800} & \frac{562618}{362800} & \frac{-662757}{362800} & \frac{694230}{362800} & \frac{-506675}{362800} & \frac{239406}{362800} & \frac{-65823}{362800} & \frac{8002}{362800} \\ \frac{13376}{28350} & \frac{38762}{28350} & \frac{-32823}{28350} & \frac{34730}{28350} & \frac{-25300}{28350} & \frac{11934}{28350} & \frac{-3277}{28350} & \frac{398}{28350} \\ \frac{51327}{44800} & \frac{183654}{44800} & \frac{-105381}{44800} & \frac{135450}{44800} & \frac{-98955}{44800} & \frac{46818}{44800} & \frac{-12879}{44800} & \frac{1566}{44800} \\ \frac{29976}{14175} & \frac{118432}{14175} & \frac{-46608}{14175} & \frac{86880}{14175} & \frac{-58520}{14175} & \frac{27744}{14175} & \frac{-7632}{14175} & \frac{928}{14175} \\ \frac{2451875}{145152} & \frac{2052250}{145152} & \frac{-577125}{145152} & \frac{1603750}{145152} & \frac{-891875}{145152} & \frac{456750}{145152} & \frac{-125375}{145152} & \frac{15250}{145152} \\ \frac{6912}{1400} & \frac{30024}{1400} & \frac{-6156}{1400} & \frac{24840}{1400} & \frac{-10800}{1400} & \frac{7128}{1400} & \frac{-1764}{1400} & \frac{216}{1400} \\ \frac{3520209}{518400} & \frac{15697738}{518400} & \frac{-2369787}{518400} & \frac{13613670}{518400} & \frac{-4621925}{518400} & \frac{4336206}{518400} & \frac{-655473}{518400} & \frac{114562}{518400} \end{pmatrix}$$

$$\begin{pmatrix} y''_{n+1} \\ y''_{n+2} \\ y''_{n+3} \\ y''_{n+4} \\ y''_{n+5} \\ y''_{n+6} \\ y''_{n+7} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} y''_n + \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{pmatrix} h y'''_n + h^2 M \begin{pmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \end{pmatrix}$$

for

$$M = \begin{pmatrix} \frac{308455}{1344780} & \frac{704619}{1344780} & \frac{-759771}{1344780} & \frac{785218}{1344780} & \frac{-569816}{1344780} & \frac{268387}{1344780} & \frac{-73642}{1344780} & \frac{8940}{1344780} \\ \frac{14939}{28350} & \frac{55642}{28350} & \frac{-34986}{28350} & \frac{39950}{28350} & \frac{-29405}{28350} & \frac{13926}{28350} & \frac{-3832}{28350} & \frac{466}{28350} \\ \frac{496773}{604800} & \frac{2113614}{604800} & \frac{-650997}{604800} & \frac{1394820}{604800} & \frac{-985365}{604800} & \frac{465102}{604800} & \frac{-127899}{604800} & \frac{15552}{604800} \\ \frac{15824}{14175} & \frac{71152}{14175} & \frac{-11496}{14175} & \frac{56720}{14175} & \frac{-29960}{14175} & \frac{14736}{14175} & \frac{-4072}{14175} & \frac{496}{14175} \\ \frac{102425}{72576} & \frac{475000}{72576} & \frac{-40125}{72576} & \frac{421250}{72576} & \frac{-130625}{72576} & \frac{102900}{72576} & \frac{-26875}{72576} & \frac{3250}{72576} \\ \frac{597}{350} & \frac{2826}{350} & \frac{-108}{350} & \frac{2670}{350} & \frac{-495}{350} & \frac{918}{350} & \frac{-126}{350} & \frac{18}{350} \\ \frac{86506}{43182} & \frac{413600}{43182} & \frac{1200}{43182} & \frac{400000}{43182} & \frac{-34000}{43182} & \frac{160800}{43182} & \frac{24400}{43182} & \frac{5453}{43182} \end{pmatrix}$$

$$\begin{pmatrix} y'''_{n+1} \\ y'''_{n+2} \\ y'''_{n+3} \\ y'''_{n+4} \\ y'''_{n+5} \\ y'''_{n+6} \\ y'''_{n+7} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} y'''_n + hN \begin{pmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \end{pmatrix}$$

for

$$N=\begin{pmatrix} \frac{36799}{120960} & \frac{139849}{120960} & \frac{-121797}{120960} & \frac{123133}{120960} & \frac{-88547}{120960} & \frac{41499}{120960} & \frac{-11351}{120960} & \frac{1375}{120960} \\ \frac{5535}{18900} & \frac{29320}{18900} & \frac{-3195}{18900} & \frac{12240}{18900} & \frac{-9635}{18900} & \frac{4680}{18900} & \frac{-1305}{18900} & \frac{160}{18900} \\ \frac{6625}{22400} & \frac{33975}{22400} & \frac{6885}{22400} & \frac{29635}{22400} & \frac{-15165}{22400} & \frac{6885}{22400} & \frac{-1865}{22400} & \frac{225}{22400} \\ \frac{278}{945} & \frac{1448}{945} & \frac{216}{945} & \frac{1784}{945} & \frac{-106}{945} & \frac{216}{945} & \frac{-64}{945} & \frac{8}{945} \\ \frac{7155}{24192} & \frac{36725}{24192} & \frac{6975}{24192} & \frac{41625}{24192} & \frac{13625}{24192} & \frac{17055}{24192} & \frac{-2475}{24192} & \frac{275}{24192} \\ \frac{41}{140} & \frac{216}{140} & \frac{27}{140} & \frac{272}{140} & \frac{27}{140} & \frac{216}{140} & \frac{41}{140} & 0 \\ \frac{5257}{17280} & \frac{25039}{17280} & \frac{9261}{17280} & \frac{20923}{17280} & \frac{20923}{17280} & \frac{9261}{17280} & \frac{25039}{17280} & \frac{5257}{17280} \end{pmatrix}$$

3 Analysis of the Properties of the Block Method

3.1 Order of the Method

The linear operator associated with Eq. (6) can be defined as

$$L\{y(x):h\} = A^0 Y_N - A' Y_{N-1} - hA'' Y'_{N-1} - h^2 B' Y''_{N-1} - h^3 B'' Y'''_{N-1}$$
$$-h^4 (E^0 F_N + E' F_{N-1}) \qquad (7)$$

Eq. (7) is expanded in Taylor series, which gives

$$L[y(x):h] = C_0 y(x) + C_1 hy'(x) + \cdots + C_p h^{(p)} y^{(p)}(x) + C_{p+1} h^{(p+1)} y^{(p+1)}(x) + \cdots$$

The Eq. (6) and the associated linear operator are said to have order p if $C_0 = C_1 = C_2 = ... = C_p = C_{p+1} = C_{p+2} = C_{p+3} 0, C_{p+4} \neq 0$. Therefore, our method's Eq. (6) has order (p) $[8,8,8,8,8,8,8]^T$ with error constants

$C_{p+4} = [\frac{-40}{90159}, \frac{-44}{6727}, \frac{-73}{2765}, \frac{-53}{781}, \frac{-454}{3269}, \frac{-332}{1343}, \frac{-317}{791}]^T$. . $C_{p+4} h^{p+4} y^{(p+4)}(x_n)$ is known as the principal local truncation error at point x_n.

3.2 Zero Stability of the Method

Block Eq. (6) is said to be zero-stable if the roots $z_s = 1, 2, ..., N$ of the first characteristic polynomial $\rho(z) = \det(zA^0 - A')$ satisfies $|z| \leq 1$ and the root

$|z|=1$ has multiplicity not greater than the order of the differential equation, which is 4.

Now, $\rho(z)=\det(zA^0-A')=0$ implies that $\rho(z)=z^6(z-1)$. Hence $z=0,0,0,0,0,0,1$. Therefore, the new method's Eq. (6) is convergent because it is zero-stable and has order greater than one.

4 Numerical Experiment

The accuracy of the new method is examined by solving the following differential problems.

Problem 1: $y^{iv}=x,\ \ y(0)=0, y'(0)=1, y''(0)=y'''(0)=0, h=0.1$

Exact solution: $y(x)=\dfrac{x^5}{120}+x$

Problem 2: $y^{iv}+y''=0,\ \ 0\le x\le\dfrac{\pi}{2},\ \ y(0)=0,\ y'(0)=\dfrac{-1.1}{72-50\pi},$

$y''(0)=\dfrac{1}{144-100\pi},\ y'''(0)=\dfrac{1.2}{144-100\pi}, h=0.01$

Exact solution: $y(x)=\dfrac{1-x-\cos x-1.2\sin x}{144-100\pi}$

Problem 3: $y^{iv}=(x^4+14x^3+49x^2+32x-12)e^x, y(0)=y'(0)=0,$

$y''(0)=2, y'''(0)=-6,\ \ 0\le x\le 1.$

Exact Solution: $y(x)=x^2(1-x)^2e^x$

Problem 4: $y^{iv}=y\ \ y(0)=y'(0)=y''(0)=y'''(0)=1,\ \ 0\le x\le 1$

Exact Solution: $y(x)=e^x$

The results generated after solving the above problems are shown in Tables 1–4.

The following notations are used in Tables 3 and 4:

S2PEB: Sequential implementation of the 2-Point Explicit Block Method.
P2PEB: Parallel implementation of the 2-Point Explicit Block Method.
S3PEB: Sequential implementation of the 3-Point Explicit Block Method.
P3PEB: Parallel implementation of the 3-Point Explicit Block Method.

Table 1 Comparison of new method with [13] and [14] for solving problem 1.

x	Exact Solution	Computed Solution	Error in [13], k = 6	Error in [14], k = 6	Error in New Method, k = 7
0.1	0.100000083333333340	0.100000083332331250	7.000024E-10	1.66666667E-10	1.002087E-12
0.2	0.200002666666666690	0.200002666666666690	8.9999912-10	3.33333305E-10	0.000000E+00
0.3	0.300020250000000040	0.300020250000000040	2.599993E-09	5.99999994E-10	0.000000E+00
0.4	0.400085333333333350	0.400085333333333350	5.100033E-09	7.66666675E-10	0.000000E+00
0.5	0.500260416666666650	0.500260416665664560	7.799979E-09	9.33333300E-10	1.002087E-12
0.6	0.600648000000000070	0.600647999997244160	1.180009E-08	1.10000009E-09	2.755907E-12
0.7	0.701400583333333440	0.701400583329826130	1.240003E-08	1.27166666E-09	3.507306E-12
0.8	0.802730666666666700	0.802730666663159400	1.410006E-08	1.45333334E-09	3.507306E-12
0.9	0.904920750000000050	0.904920749995824500	1.880000E-08	1.64999991E-09	4.175549E-12
1.0	1.008333333333333300	1.008333333328573300	1.008335E-08	1.87666660E-09	4.759970E-12

Table 2 Comparison of new method with [10] and [12] for solving problem 2.

x	Exact Solution	Computed Solution	Error in [10], k = 5	Error in [12], k = 5	Error in New Method k = 7
0.1	0.000128995622844037	0.000128995622844037	6.5052E-19	4.8355E-17	4.607859E-20
0.2	0.000257396543210136	0.000257396543210136	1.3010E-18	1.3933E-16	5.421011E-20
0.3	0.000385195797911474	0.000385195797911474	4.7704E-18	6.6893E-16	2.710505E-19
0.4	0.000512386483927295	0.000512386483927295	1.7347E-17	2.0129E-15	1.084202E-19
0.5	0.000764914842785370	0.000764914842785371	4.3368E-17	4.6736E-15	4.336809E-19
0.6	0.000764914842785370	0.000764914842785371	9.5409E-17	9.1874E-15	8.673617E-19
0.7	0.000890239016598606	0.000890239016598607	1.8127E-16	1.6069E-14	1.084202E-18
0.8	0.001014927625018177	0.001014927625018179	3.1571E-16	2.5407E-14	1.734723E-18
0.9	0.001138974076085363	0.001138974076085365	5.1868E-16	3.8108E-14	2.818926E-18
1.0	0.001262371842056641	0.001262371842056644	8.0491E-16	5.4051E-14	3.469447E-18

Table 3 Comparison of new method with [7] for solving problem 3.

h-values	New Method	Omar In [7]	Number of Steps	Error in New Method, k = 7	Error in [7] k = 8
10^{-2}	7-Step Method	S2PEB	54	3.547029E-11	1.00778E-02
		P2PEB	54	3.547029E-11	1.00778E-02
		S3PEB	39	1.250555E-12	1.00778E-02
		P3PEB	39	1.250555E-12	1.00778E-02
10^{-3}	7-Step Method	S2PEB	504	1.141416E-10	1.00778E-03
		P2PEB	504	1.141416E-10	1.00778E-02
		S3PEB	339	1.762146E-12	1.00778E-03
		P3PEB	339	1.762146E-12	1.00778E-02
10^{-4}	7-Step Method	S2PEB	5004	1.439730E-09	1.00008E-04
		P2PEB	5004	1.439730E-09	1.00008E-04
		S3PEB	3339	3.311129E-12	1.00008E-04
		P3PEB	3339	3.311129E-12	1.00008E-04
10^{-5}	7-Step Method	S2PEB	50004	2.466095E-09	1.00001E-05
		P2PEB	50004	2.466095E-09	1.00001E-05
		S3PEB	33339	9.124790E-11	1.00001E-05
		P3PEB	33339	9.124790E-11	1.00001E-05

Table 4 Comparison of new method with [7] for solving problem 4.

h-values	New Method	Omar In [7]	Number of Steps	Error in New Method, k = 7	Error in [7] k = 8
10^{-2}	7-Step Method	S2PEB	54	3.725589E-10	8.37112E-04
		P2PEB	54	3.725589E-10	8.37112E-04
		S3PEB	39	1.074216E-10	8.37105E-04
		P3PEB	39	1.074216E-10	8.37105E-04
10^{-3}	7-Step Method	S2PEB	504	3.765876E-13	8.34604E-05
		P2PEB	504	3.765876E-13	8.34604E-05
		S3PEB	339	6.750156E-14	8.34604E-05
		P3PEB	339	6.750156E-14	8.34604E-05
10^{-4}	7-Step Method	S2PEB	5004	7.297274E-12	8.34353E-06
		P2PEB	5004	7.297274E-12	8.34353E-06
		S3PEB	3339	2.362555E-13	8.34353E-06
		P3PEB	3339	2.362555E-13	8.34353E-06
10^{-5}	7-Step Method	S2PEB	50004	2.202682E-11	8.34326E-07
		P2PEB	50004	2.202682E-11	8.34326E-07
		S3PEB	33339	4.297007E-12	8.34330E-07
		P3PEB	33339	4.297007E-12	8.34330E-07

5 Conclusion

A seven-step block method for the solution of fourth order ODEs is proposed in this paper. The new method was used to solve fourth order IVPs of ODEs. The numerical results were compared with the existing methods. The new method performed better than the methods in [10,12-14], which employed 5 and 6 steps (refer to Tables 1 and 2). This implies that better accuracy can be achieved when step number k is increased. The accuracy of the new method was also found to be better than the method in [7], which was developed through numerical integration using 8 steps (refer to Tables 3 and 4). Thus, based on the numerical results, the new method outperformed the existing methods in terms of accuracy and should be considered as a viable alternative for directly solving fourth order initial value problems.

References

[1] Awari, Y.S. & Abada, A.A., *A Class of Seven Point Zero Stable Continuous Block Method for Solution of Second Order Ordinary Differential Equation*, International Journal and Mathematics and Statistics Invention (IJMSI), **2**(1), pp. 47-54, 2014.

[2] Anake, T.A., Awoyemi D.O. & Adesanya, A.O., *One-Step Implicit Hybrid Block Method for the Direct Solution of General Second Order Ordinary Differential Equations*, IEANG International Journal of Applied Mathematics, **42**(4), IJAM_42_4_04, 2012.

[3] Bun, R.A. & Vasil'yer., *A Numerical Method for Solving Differential Equation of Any Orders*, Comp. Math. Phys., **32**(3), pp. 317-330, 1992.

[4] Brugnano, L. & Trigiante, D., *Solving Differential Problems by Multistep IVPs and BVP Methods,* Gordon and Breach Science Publishers, 1998.

[5] Lambert, J.D., *Computational Methods in Ordinary Differential Equations*, John Wiley & Sons Inc. 1973.

[6] Senu, N., Suleiman, M., Ismail, F. & Othman, M., *A Singly Diagonally Implicit Runge-kutta-Nystrom Method for Solving Oscillatory Problems*, IEANG International Journal of Applied Mathematics, **41**(2), IJAM _41_2_11, 2011.

[7] Omar, Z., *Parallel Block Methods for Solving Higher Order Ordinary Differential Equation Directly*. Doctor Thesis in the Faculty of Science and Environmental Studies Universiti Putra Malaysia, Unpublished, 1999.

[8] Omar, Z. & Kuboye, J.O., *Computation of an Accurate Implicit Block Method for Solving Third Order Ordinary Differential Equations Directly*, Global Journal of Pure and Applied Mathematics, **11**(1), pp. 177-186, 2015.

[9] Omar, Z. & Kuboye, J.O., *Developing Block Method of Order Seven for Solving Third Order Ordinary Differential Equations Directly using Multistep Collocation Approach*. International Journal of Applied Mathematics and Statistics™, **53**(3), pp. 165-173, 2015.

[10] Adesanya, A.O., Momoh, A.A., Alkali, M.A. & Tahir, A., *A Five Steps Block Method for the Solution of fourth Order Ordinary Differential Equations*, International Journal of Engineering Research and Applications (IJERA), **2**(4), pp. 991-998, 2012.

[11] Butcher, J.C, *Numerical Methods for Ordinary Differential Equations*, John Wiley & Sons, New York, 2003.

[12] Kayode, S.J., *An Efficient Zero Stable Method for Fourth Order Ordinary Differential Equations*, Int. J. Math. Sci, pp. 1-10, 2008. dx.doi.org/10.1155/2008/364021

[13] Mohammed, U., *A Six Step Block Method for Solution of Fourth Order Ordinary Differential Equations*, The Pacific Journal of Science and Technology, **11**(1), pp. 259-265, 2010.

[14] Olabode, B.T., *A Six Step Scheme for the Solution of Fourth Order Ordinary Differential Equations*, The Pacific Journal of Science and Technology, **10**(1), pp. 143-148, 2009.

[15] Omar, Zurni, and Kuboye, J.O., *Computation of an Accurate Implicit Block Method for Solving Third Order Ordinary Differential Equations Directly*, Global Journal of Pure and Applied Mathematics, **11**(1), pp. 177-186, 2015.

[16] Omar, Z. & Kuboye, J.O., *Developing Block Method of Order Seven for Solving Third Order Ordinary Differential Equations Directly using Multistep Collocation Approach*. International Journal of Applied Mathematics and Statistics™, **53**(3), pp. 165-173, 2015.